Transformations

- Why use transformations?
  - Create object in convenient coordinates
  - Reuse basic shape multiple times
  - Hierarchical modeling
  - System independent
  - Virtual cameras

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$T(t_x, t_y, t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Properties of Translation

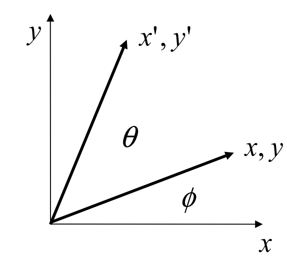
$$T(0,0,0)\,\mathbf{v} = \mathbf{v}$$

$$T(s_x, s_y, s_z)\,T(t_x, t_y, t_z)\mathbf{v} = T(s_x + t_x, s_y + t_y, s_z + t_z)\,\mathbf{v}$$

$$T(s_x, s_y, s_z)\,T(t_x, t_y, t_z)\mathbf{v} = T(t_x, t_y, t_z)\,T(s_x, s_y, s_z)\mathbf{v}$$

$$T^{-1}(t_x, t_y, t_z)\,\mathbf{v} = T(-t_x, -t_y, -t_z)\,\mathbf{v}$$

Rotations (2D)



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$x' = r\cos(\phi + \theta)$$
$$y' = r\sin(\phi + \theta)$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi + \theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

$$x' = (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$
$$y' = (r\cos\phi)\sin\theta + (r\sin\phi)\cos\theta$$

$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

Rotations 2D

- So in matrix notation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotations (3D)

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of Rotations

$$R_a(0) = I$$

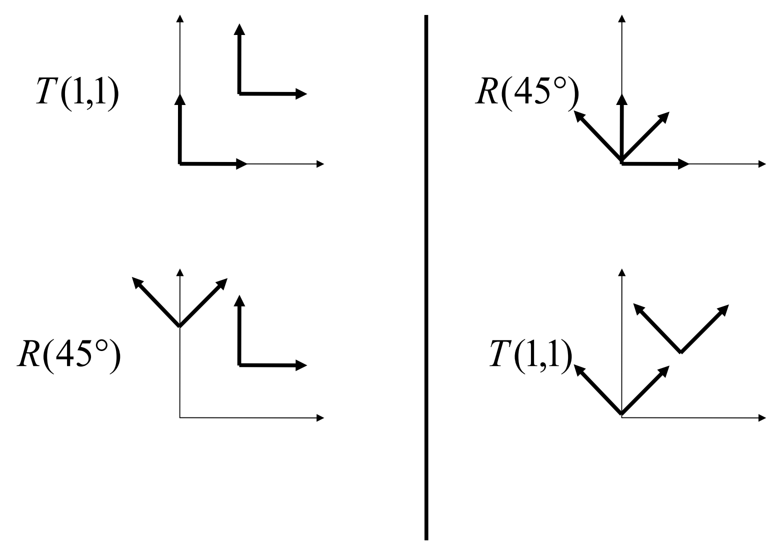
$$R_a(\theta)R_a(\phi) = R_a(\phi+\theta)$$

$$R_a(\theta)R_a(\phi) = R_a(\phi)R_a(\theta)$$

$$R_a^{-1}(\theta) = R_a(-\theta) = R_a^{T}(\theta)$$

$$R_a(\theta)R_b(\phi) \neq R_b(\phi)R_a(\theta)$$ order matters!

Combining Translation & Rotation



Combining Translation & Rotation

$$\mathbf{v}' = \mathbf{v} + T$$

$$\mathbf{v}'' = R\mathbf{v}'$$

$$\mathbf{v}'' = R(\mathbf{v} + T)$$

$$\mathbf{v}'' = R\mathbf{v} + RT$$

$$\mathbf{v}' = R\mathbf{v}$$

$$\mathbf{v}'' = \mathbf{v}' + T$$

$$\mathbf{v}'' = R\mathbf{v} + T$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \end{bmatrix}$$

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

Uniform scaling ***iff*** $s_x = s_y = s_z$

Homogeneous Coordinates

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be represented as $\begin{bmatrix} X \\ Y \\ Z \\ w \end{bmatrix}$

where $x = \dfrac{X}{w}, \quad y = \dfrac{Y}{w}, \quad z = \dfrac{Z}{w}$

Translation Revisited

$$T(t_x, t_y, t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation & Scaling Revisited

$$R_x(\theta)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$S(s_x, s_y, s_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Combining Transformations

$$\mathbf{v}' = S\mathbf{v}$$
$$\mathbf{v}'' = R\mathbf{v}' = RS\mathbf{v}$$
$$\mathbf{v}''' = T\mathbf{v}'' = TR\mathbf{v}' = TRS\mathbf{v}$$
$$\mathbf{v}''' = M\mathbf{v}$$

where $M = TRS$

Transforming Tangents

$$\mathbf{t} = \mathbf{p} - \mathbf{q}$$

$$\mathbf{t}' = \mathbf{p}' - \mathbf{q}'$$
$$= M\mathbf{p} - M\mathbf{q}$$
$$= M(\mathbf{p} - \mathbf{q})$$
$$= M\mathbf{t}$$

Transforming Normals

$$\mathbf{n}^T \mathbf{t} = 0$$
$$\mathbf{n}'^T \mathbf{t}' = 0$$
$$\mathbf{n}'^T M\mathbf{t} = 0$$
$$\mathbf{n}'^T M\mathbf{t} = \mathbf{n}^T \mathbf{t}$$
$$\mathbf{n}'^T M = \mathbf{n}^T$$
$$M^T \mathbf{n}' = \mathbf{n}$$
$$\mathbf{n}' = M^{T^{-1}} \mathbf{n} = M^{-1^T} \mathbf{n}$$