

QUALITY LOCAL REFINEMENT OF TETRAHEDRAL MESHES BASED ON 8-SUBTETRAHEDRON SUBDIVISION

Anwei Liu and Barry Joe
Department of Computing Science
University of Alberta
Edmonton, AB, Canada T6G 2H1
anwei@cs.ualberta.ca, barry@cs.ualberta.ca

April 28, 1994

ABSTRACT. Let \mathcal{T} be a tetrahedral mesh. We present a 3-D local refinement algorithm for \mathcal{T} which is mainly based on an *8-subtetrahedron subdivision* procedure, and discuss the quality of refined meshes generated by the algorithm. It is proved that any tetrahedron $\mathbf{T} \in \mathcal{T}$ produces a finite number of classes of similar tetrahedra, independent of the number of refinement levels. Furthermore, $\eta(\mathbf{T}_i^n) \geq c\eta(\mathbf{T})$, where $\mathbf{T} \in \mathcal{T}$, c is a positive constant independent of \mathcal{T} and the number of refinement levels, \mathbf{T}_i^n is any refined tetrahedron of \mathbf{T} , and η is a tetrahedron shape measure. It is also proved that local refinements on tetrahedra can be smoothly extended to their neighbors to maintain a conforming mesh. Experimental results show that the ratio of the number of tetrahedra actually refined to the number of tetrahedra chosen for refinement is bounded above by a small constant.

1. INTRODUCTION

The solution of some types of partial differential equations (PDEs) using the finite element method is an adaptive process, which in general consists of mesh generation, solution of linear equations, error estimation, and mesh refinement. In the mesh refinement step, varying element sizes are used to improve the solution of PDEs, and an efficient scheme, local refinement, is adopted to locally refine some regions where approximation errors are unacceptable. Also, the adaptive mesh refinement technique is needed in the multigrid method, in which the solution of a problem is obtained by alternatively solving the problem on several levels of coarse to fine grids. It is desirable that the adaptive refinement produces refined elements of good shape using a fast refining process. These are the goals of our 3-D local refinement algorithm (LRA).

Not many results have been published on the quality of meshes produced by 3-D local refinement algorithms (LRAs), although some 2-D quality LRAs are available [Riv84, Riv87, Riv89, BaS81, Ban90, LMZ85] and have proved to be successful in adaptive finite element analysis [Mit89]. Some 3-D LRAs are used in practice without theoretical analysis on the quality of the refined meshes [NeF91, RiL92]. In [LiJ94c], we presented an LRA for tetrahedral meshes based on a bisection procedure, which is the first theoretically guaranteed-quality LRA for tetrahedral meshes. In this

1991 *Mathematics Subject Classification.* Primary 65N50; Secondary 51M20, 52B10, 65M50.

This work was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

paper, we present a quality LRA based on an *8-subtetrahedron subdivision* procedure, which is an extension of the 2-D LRA in [BaS81, Ban90, LMZ85]. The 8-subtetrahedron subdivision may be preferable to bisection if the initial mesh is relatively coarse and needs to be refined quickly (this is also the case in the multigrid method, since the grids at two consecutive levels should in general be significantly different), because the volume of subtetrahedra in 8-subtetrahedron subdivision decreases faster than in bisection.

We consider a local refinement process that includes two major steps. First, a set S of basic elements is chosen from an existing mesh \mathcal{T} according to numerical results and error estimates from previous computations. Next, each element in S is refined, and a procedure is needed to keep the final mesh conforming, where a *conforming* mesh is one in which the intersection of any two tetrahedra $\mathbf{T}_1, \mathbf{T}_2$ of the mesh is either a common face of \mathbf{T}_1 and \mathbf{T}_2 , or a common edge, or a common vertex, or empty.

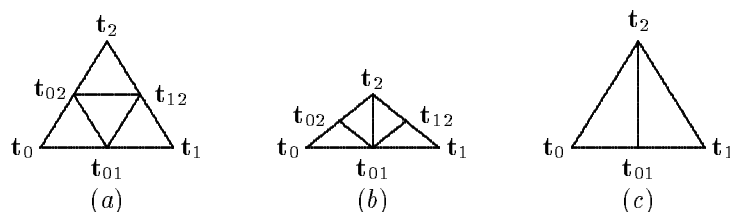


FIGURE 1. Illustration for 2-D local refinement in PLTMG; $\mathbf{t}_{ij} = (\mathbf{t}_i + \mathbf{t}_j)/2$, $i < j$. (a) Regular refinement; no interior angle is greater than $\pi/2$. (b) Regular refinement; one interior angle is greater than $\pi/2$. (c) Irregular refinement.

The basic idea in [Ban90] is as follows. Normally, a triangle is regularly refined by dividing it into 4 similar triangles, as illustrated in Figure 1a. If the triangle has an interior angle greater than $\pi/2$, then it is regularly refined into two similar triangles, and two geometrically better triangles, as illustrated in Figure 1b. At the boundary of a refined region, it is necessary to divide a triangle into only 2 “green” triangles by inserting a “green” edge, as illustrated in Figure 1c. Refinement into green triangles is done only as a temporary measure; at each step of the adaptive process, if the green triangles need to be further refined, then the green edges of these triangles are removed and the parent elements are regularly refined. Obviously, the degradation of geometry in this approach is bounded, since the green triangles are never further refined.

Likewise, in 3-D, we want to use a regular refinement called *8-subtetrahedron subdivision*, which is defined as follows. Suppose each triangular face of a tetrahedron \mathbf{T} is refined into four similar subtriangles by connecting the midpoints of the edges (see Figure 1a) as shown in Figure 2. Then we obtain four similar subtetrahedra at the four corners and an octahedron in the interior. By adding an interior edge, called the *centre edge*, $\mathbf{t}_{02}\mathbf{t}_{13}$ say, in the middle of the octahedron, \mathbf{T} is subdivided into eight subtetrahedra. We use SUB_8 to denote the 8-subtetrahedron subdivision described here.

Given a tetrahedron \mathbf{T} , SUB_8 can be performed to \mathbf{T} and its subtetrahedra repeatedly to produce a sequence of meshes. Note that the centre edge in SUB_8 can be any one of the three choices (e.g., $\mathbf{t}_{01}\mathbf{t}_{23}$, $\mathbf{t}_{02}\mathbf{t}_{13}$, or $\mathbf{t}_{03}\mathbf{t}_{12}$). Different strategies for choosing centre edges will produce substantially different meshes in terms of the quality of refined meshes. In Section 2, we describe a refinement procedure, i.e., a way of selecting the centre edges, such that (a) iteratively applying SUB_8 to any

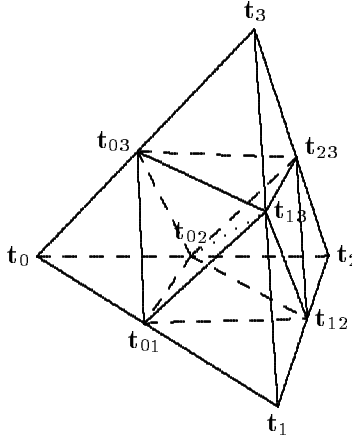


FIGURE 2. Illustration for the regular refinement (8-subtetrahedron subdivision) SUB_8 ; $\mathbf{t}_{ij} = (\mathbf{t}_i + \mathbf{t}_j)/2, i < j$; eight subtetrahedra are $\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_{01}, \mathbf{t}_{02}, \mathbf{t}_{03})$, $\mathbf{T}_2^1(\mathbf{t}_{01}, \mathbf{t}_1, \mathbf{t}_{12}, \mathbf{t}_{13})$, $\mathbf{T}_3^1(\mathbf{t}_{02}, \mathbf{t}_{12}, \mathbf{t}_2, \mathbf{t}_{23})$, $\mathbf{T}_4^1(\mathbf{t}_{03}, \mathbf{t}_{13}, \mathbf{t}_{23}, \mathbf{t}_3)$, $\mathbf{T}_5^1(\mathbf{t}_{01}, \mathbf{t}_{13}, \mathbf{t}_{03}, \mathbf{t}_{02})$, $\mathbf{T}_6^1(\mathbf{t}_{01}, \mathbf{t}_{12}, \mathbf{t}_{13}, \mathbf{t}_{02})$, $\mathbf{T}_7^1(\mathbf{t}_{23}, \mathbf{t}_{02}, \mathbf{t}_{13}, \mathbf{t}_{12})$, and $\mathbf{T}_8^1(\mathbf{t}_{23}, \mathbf{t}_{02}, \mathbf{t}_{13}, \mathbf{t}_{03})$.

tetrahedron \mathbf{T} produces at most 24 classes of similar tetrahedra, and (b) $\eta(\mathbf{T}_i^n) \geq 0.5\eta(\mathbf{T})$, where \mathbf{T}_i^n is any refined tetrahedron of \mathbf{T} and η is the mean ratio [LiJ94a], a tetrahedron shape measure. In Section 3, we present an LRA extensively based on SUB_8 , and some properties of the algorithm are established. In Section 4, experimental results are provided.

2. SUBDIVISION OF A SINGLE TETRAHEDRON

In this section, we are mainly concerned about the quality of meshes produced by repeatedly performing SUB_8 to a single tetrahedron \mathbf{T} without considering the conformity of the refined meshes (which will be discussed in the next section). We first give some notation. For any (nondegenerate) tetrahedron $\mathbf{T}(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ with $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ as the coordinates of the four vertices in the form of column vectors in E^3 , define the 3 by 3 nonsingular matrix $T = [\mathbf{t}_1 - \mathbf{t}_0, \mathbf{t}_2 - \mathbf{t}_0, \mathbf{t}_3 - \mathbf{t}_0]$. Note that the matrix has the same name as the tetrahedron but italic font is used instead of bold font, and T depends on the ordering of vertices of \mathbf{T} . For any two tetrahedra $\mathbf{S}(\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ and $\mathbf{T}(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, define the 3 by 3 matrices $M(\mathbf{S}, \mathbf{T}) = TS^{-1}$ and $A(\mathbf{S}, \mathbf{T}) = M^T(\mathbf{S}, \mathbf{T})M(\mathbf{S}, \mathbf{T})$. Note that M and A also depend on the ordering of tetrahedron vertices, and M is the matrix involved in the affine transformation from \mathbf{S} to \mathbf{T} such that $\mathbf{t}_i = M(\mathbf{S}, \mathbf{T})\mathbf{s}_i + \mathbf{b}$, $0 \leq i \leq 3$, where $\mathbf{b} = \mathbf{t}_0 - M(\mathbf{S}, \mathbf{T})\mathbf{s}_0$.

Tetrahedron shape measure η , defined in [LiJ94a], is used in analyzing the quality of meshes generated by our algorithm. The properties of η and the relationship between η and other shape measures can be found in [LiJ94a, LiJ94b]. The following two formulae are given in [LiJ94a], and often used in the remainder of this section.

$$(1) \quad \eta(\mathbf{T}) = 3\sqrt[3]{\lambda_1\lambda_2\lambda_3}/(\lambda_1 + \lambda_2 + \lambda_3) = 3\sqrt[3]{\det(A(\mathbf{R}, \mathbf{T}))/\text{trace}(A(\mathbf{R}, \mathbf{T}))},$$

where λ_1, λ_2 , and λ_3 are the eigenvalues of matrix $A(\mathbf{R}, \mathbf{T})$, \mathbf{R} is a regular tetrahedron with the same volume as \mathbf{T} , and η is independent of the ordering of tetrahedron vertices and of the vertex coordinates of \mathbf{R} .

$$(2) \quad \eta(\mathbf{T}) = 12(3v)^{2/3} / \sum_{0 \leq i < j \leq 3} l_{ij}^2,$$

where v is the volume of \mathbf{T} and the l_{ij} are the lengths of the edges of \mathbf{T} .

As mentioned in the previous section, the centre edge in SUB_8 is the edge connecting a pair of opposite edges, called the *base edges*. That is, the centre edge is the edge joining the midpoints of a pair of base edges. We say that a subtetrahedron has the same *subdivision pattern* as its direct parent if its base edges include either the centre edge or a half of a base edge of its parent, e.g., in Figure 2, if the base edges of $\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_{01}, \mathbf{t}_{02}, \mathbf{t}_{03})$ are $\mathbf{t}_0\mathbf{t}_{02}$ and $\mathbf{t}_{01}\mathbf{t}_{03}$, then $\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_{01}, \mathbf{t}_{02}, \mathbf{t}_{03})$ has the same subdivision pattern as $\mathbf{T}(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, since $\mathbf{t}_0\mathbf{t}_{02}$ is a half of the base edge $\mathbf{t}_0\mathbf{t}_2$ of \mathbf{T} . It can be easily verified that if a subtetrahedron has the same subdivision pattern as its parent, then its two base edges and centre edge are uniquely determined. Now we describe a procedure SUBDIV for iteratively applying SUB_8 to \mathbf{T} .

Algorithm SUBDIV

- (1) Subdivide $\mathbf{T} = \mathbf{T}_1^0$ into eight subtetrahedra using SUB_8 (see Figure 2); the centre edge can be any of the three choices, i.e., $\mathbf{t}_{01}\mathbf{t}_{23}$, $\mathbf{t}_{02}\mathbf{t}_{13}$, or $\mathbf{t}_{03}\mathbf{t}_{12}$; label the eight subtetrahedra by \mathbf{T}_i^1 , $1 \leq i \leq 8$, and let \mathbf{T}_i^1 have the same subdivision pattern as \mathbf{T} ; $n := 1$.
- (2) Subdivide \mathbf{T}_i^n , $1 \leq i \leq 8^n$, using SUB_8 , and let its subtetrahedra have the same subdivision pattern as \mathbf{T}_i^n ; label the resulting subtetrahedra by \mathbf{T}_i^{n+1} , $1 \leq i \leq 8^{n+1}$.
- (3) $n := n + 1$; repeat (2) or terminate the subdivision.

The superscript n in \mathbf{T}_i^n denotes the level of subdivision. We assume that each subtetrahedron is subdivided to the same level (this constraint will be removed in the next section). In order to study the quality of the mesh produced by SUBDIV (which is a main procedure in our LRA in the next section), we first describe SUBDIV by another equivalent procedure, which is relatively easy to use in analyzing the shape of the refined meshes. To this end, we introduce a canonical tetrahedron \mathbf{P} called a *rhombic tetrahedron* [MoW90], shown in Figure 3, which is used in an alternative description of SUBDIV . \mathbf{P} has two longer edges of the same length and four shorter edges of the same length. The most important property of \mathbf{P} is given in the following lemma, which is proved in [MoW90].

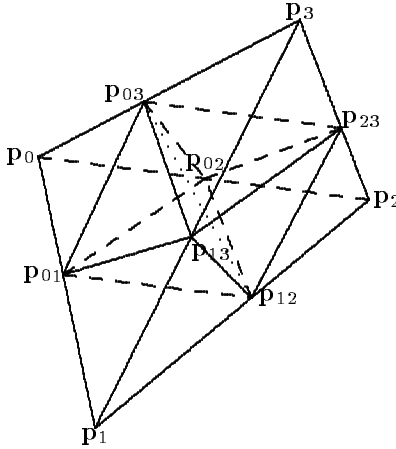


FIGURE 3. Rhombic tetrahedron $\mathbf{P}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ with $|\mathbf{p}_0\mathbf{p}_1| = |\mathbf{p}_2\mathbf{p}_3| = |\mathbf{p}_1\mathbf{p}_3| = |\mathbf{p}_0\mathbf{p}_2| = m$, $|\mathbf{p}_0\mathbf{p}_3| = |\mathbf{p}_1\mathbf{p}_2| = 2m/\sqrt{3}$; $\mathbf{p}_{ij} = (\mathbf{p}_i + \mathbf{p}_j)/2$. The coordinates are $\mathbf{p}_0(-\sqrt{2}m/\sqrt{3}, 0, 0)$, $\mathbf{p}_1(0, -m/\sqrt{3}, 0)$, $\mathbf{p}_2(0, m/\sqrt{3}, 0)$, $\mathbf{p}_3(0, 0, \sqrt{2}m/\sqrt{3})$, $\mathbf{p}_{01}(-\sqrt{6}m/6, -\sqrt{3}m/6, 0)$, $\mathbf{p}_{02}(-\sqrt{6}m/6, \sqrt{3}m/6, 0)$, $\mathbf{p}_{03}(-\sqrt{6}m/6, 0, \sqrt{6}m/6)$, $\mathbf{p}_{12}(0, 0, 0)$, $\mathbf{p}_{13}(0, -\sqrt{3}m/6, \sqrt{6}m/6)$, and $\mathbf{p}_{23}(0, \sqrt{3}m/6, \sqrt{6}m/6)$.

Lemma 1. *If SUB_8 is applied to \mathbf{P} with the centre edge connecting its longest edges $\mathbf{p}_0\mathbf{p}_3$ and $\mathbf{p}_1\mathbf{p}_2$, then the eight subtetrahedra $\mathbf{P}_1^1(\mathbf{p}_0, \mathbf{p}_{01}, \mathbf{p}_{02}, \mathbf{p}_{03})$, $\mathbf{P}_2^1(\mathbf{p}_{01}, \mathbf{p}_1, \mathbf{p}_{12}, \mathbf{p}_{13})$, $\mathbf{P}_3^1(\mathbf{p}_{02}, \mathbf{p}_{12}, \mathbf{p}_2, \mathbf{p}_{23})$, $\mathbf{P}_4^1(\mathbf{p}_{03}, \mathbf{p}_{13}, \mathbf{p}_{23}, \mathbf{p}_3)$, $\mathbf{P}_5^1(\mathbf{p}_{01}, \mathbf{p}_{12}, \mathbf{p}_{03}, \mathbf{p}_{13})$, $\mathbf{P}_6^1(\mathbf{p}_{23}, \mathbf{p}_{03}, \mathbf{p}_{12}, \mathbf{p}_{13})$, $\mathbf{P}_7^1(\mathbf{p}_{01}, \mathbf{p}_{03}, \mathbf{p}_{12}, \mathbf{p}_{02})$, and $\mathbf{P}_8^1(\mathbf{p}_{23}, \mathbf{p}_{12}, \mathbf{p}_{03}, \mathbf{p}_{02})$ are all similar to the original tetrahedron $\mathbf{P}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ (see Figure 3).*

When SUB_8 is applied to \mathbf{P} , the base edges of \mathbf{P} are its longest edges ($\mathbf{p}_0\mathbf{p}_3$, $\mathbf{p}_1\mathbf{p}_2$). With each subtetrahedron having the same subdivision pattern as its parent, it is obvious that the base edges of \mathbf{P}_i^1 , $1 \leq i \leq 4$, are its longest edges. Since $|\mathbf{p}_{03}\mathbf{p}_{12}| = |\mathbf{p}_0\mathbf{p}_3|/2 = |\mathbf{p}_1\mathbf{p}_2|/2$, the base edges of \mathbf{P}_i^1 , $5 \leq i \leq 8$, are also its longest edges. Therefore, by Lemma 1 and induction, the base edges of any refined subtetrahedron of \mathbf{P} produced by SUB_8 are always its two longest edges.

Let \mathbf{P} be the canonical tetrahedron in Figure 3 such that \mathbf{T} and \mathbf{P} have the same volume. We describe another procedure TRANSUB which uses an affine transformation, and prove that SUBDIV and TRANSUB are equivalent.

Algorithm TRANSUB

- (1) Transform \mathbf{T} to \mathbf{P} by an affine transformation $\mathbf{y} = M^{-1}(\mathbf{P}, \mathbf{T})\mathbf{x} + \mathbf{b}_0$, with the constraint that the two base edges of \mathbf{T} in step (1) of SUBDIV are transformed into the pair of edges with the longest length in \mathbf{P} , i.e., $\mathbf{p}_0\mathbf{p}_3$ and $\mathbf{p}_1\mathbf{p}_2$ in Figure 3.
- (2) Iteratively subdivide \mathbf{P} to some level, with each subtetrahedron having the same subdivision pattern as its parent (i.e. the two base edges are the two longest edges).
- (3) Transform all subtetrahedra in \mathbf{P} back to subtetrahedra in \mathbf{T} using the inverse affine transformation $\mathbf{y} = M(\mathbf{P}, \mathbf{T})\mathbf{x} + \mathbf{b}_1$.

Theorem 1. *For any tetrahedron \mathbf{T} , if each subtetrahedron is refined to the same level, SUBDIV and TRANSUB produce the same set of refined tetrahedra.*

Proof. The theorem can be established by using the fact that the midpoint of a line segment is transformed into the midpoint of the corresponding line segment under an affine transformation. \square

We now use TRANSUB to establish the main properties of meshes produced by SUBDIV.

Theorem 2. *There are at most 24 classes of similar tetrahedra produced by SUBDIV in all the refined subtetrahedra of \mathbf{T} .*

Proof. We define two tetrahedra to be in the same equivalence class if one can be transformed into the other by translation and uniform scaling (i.e., the scale factors for the three coordinate axes are the same). So any two tetrahedra in the same equivalence class are similar to each other after any affine transformation. Let \mathbf{P}_i^n be any subtetrahedron of \mathbf{P} at level n , $n \geq 0$. In order to prove the theorem, it suffices to prove that all subtetrahedra \mathbf{P}_i^n generated by TRANSUB are in at most 24 different equivalence classes.

After one level of subdivision, by Lemma 1, all the 8 subtetrahedra, denoted by \mathbf{P}_i^1 , $1 \leq i \leq 8$, are similar to \mathbf{P} . Let these tetrahedra be labeled as in Lemma 1. Let $M_i = M(\mathbf{P}, \mathbf{P}_i^1)$. From the coordinates shown in Figure 3, we obtain $M_1 = M_2 = M_3 = M_4 = 1/2 I$, where I is the identity matrix, and

$$M_5 = \frac{1}{4} \begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}, \quad M_6 = \frac{1}{4} \begin{bmatrix} -1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & -\sqrt{2} \\ -1 & -\sqrt{2} & 1 \end{bmatrix},$$

$$M_7 = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & -1 \end{bmatrix}, \quad M_8 = \frac{1}{4} \begin{bmatrix} -1 & -\sqrt{2} & -1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & -1 \end{bmatrix}.$$

Let

$$\text{diag}(a_1, a_2, a_3) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad \text{skew}(a_1, a_2, a_3) = \begin{bmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_3 & 0 & 0 \end{bmatrix}.$$

The following equations are derived by straightforward computation.

$$\begin{aligned} M_7 &= \text{skew}(-1, -1, -1)M_6, \quad M_8 = \text{skew}(-1, -1, -1)M_5, \\ M_5M_6 &= 1/4\text{skew}(1, 1, -1)I, \quad M_6M_5 = 1/2\text{skew}(-1, 1, 1)M_6, \\ M_5M_7 &= 1/4\text{skew}(-1, 1, 1)I, \quad M_7M_5 = 1/2\text{diag}(-1, -1, 1)M_6, \\ M_5M_8 &= -1/4\text{skew}(1, 1, 1)I, \quad M_8M_5 = 1/4\text{diag}(-1, 1, -1)I, \\ M_6M_7 &= 1/2\text{diag}(1, -1, -1)M_5, \quad M_7M_6 = 1/2\text{diag}(-1, -1, 1)M_5, \\ M_6M_8 &= 1/2\text{diag}(1, -1, -1)M_6, \quad M_8M_6 = 1/4\text{diag}(1, -1, -1)I, \\ M_7M_8 &= 1/2\text{skew}(1, 1, -1)M_6, \quad M_8M_7 = 1/4\text{diag}(-1, -1, 1)I, \\ M_5^2 &= 1/4\text{skew}(1, -1, 1)I, \quad M_6^2 = 1/2\text{skew}(-1, 1, 1)M_5, \\ M_7^2 &= 1/2\text{skew}(1, 1, -1)M_5, \quad M_8^2 = 1/4I. \end{aligned}$$

After 2 levels of subdivision, $M(\mathbf{P}, \mathbf{P}_i^2) = M(\mathbf{P}_j^1, \mathbf{P}_i^2)M(\mathbf{P}, \mathbf{P}_j^1)$ where \mathbf{P}_i^2 is a subtetrahedron of \mathbf{P}_j^1 . Using the notation at the beginning of this section and a suitable ordering of vertices of \mathbf{P}_i^2 , it follows that $P_i^2 = M_j P_l^1$ for some l . So,

$$M(\mathbf{P}, \mathbf{P}_i^2) = P_i^2 P^{-1} = M_j P_l^1 P^{-1} = M_j M_l,$$

since $P_l^1 = M_l P$. By induction, after k levels of subdivision, for each subtetrahedron \mathbf{P}_i^k with a suitable ordering of vertices \mathbf{p}_{ir}^k , we have

$$\mathbf{p}_{ir}^k = M_{l_1} M_{l_2} \cdots M_{l_k} \mathbf{p}_r + \mathbf{b}_i^k, \quad 0 \leq r \leq 3,$$

where each M_{l_m} , $1 \leq m \leq k$, is one of the M_i , $1 \leq i \leq 8$. Let \mathcal{S} be the set of all diagonal and skew diagonal 3 by 3 matrices with elements 1 or -1 . Obviously, \mathcal{S} is closed under matrix multiplication and $|\mathcal{S}| = 2^3 + 2^3 = 16$. By the above equations, $M_{l_1} M_{l_2} \cdots M_{l_k} = fDM$, where f is a scale factor, D is an element of \mathcal{S} , and M is I , M_5 , or M_6 . Note that D and $-D$ can be considered to be the same matrix of \mathcal{S} due to the factor f . Therefore the number of different equivalence classes is $\leq 3 \times 8 = 24$. Hence the total number of classes of similar tetrahedra in all refined subtetrahedra of \mathbf{T} is finite and bounded above by 24. \square

Theorem 2 implies that the shape of subtetrahedra does not deteriorate arbitrarily. Moreover, the following theorem shows that the shape of the subtetrahedra are at least half as good as the shape of the very first tetrahedron \mathbf{T} in terms of the shape measure η .

Theorem 3. *For any refined subtetrahedron \mathbf{T}_i^n of \mathbf{T} , produced by SUBDIV,*

$$(3) \quad \eta(\mathbf{T}_i^n) \geq 0.5\eta(\mathbf{T}),$$

and the lower bound is tight.

Proof. Let $M(\mathbf{P}, \mathbf{T})$ and $M(\mathbf{R}, \mathbf{P})$ be the two matrices involved in the affine transformations from \mathbf{P} to \mathbf{T} and \mathbf{R} to \mathbf{P} , respectively. Using the notation given at the beginning of this section,

$$T = M(\mathbf{P}, \mathbf{T})P = M(\mathbf{P}, \mathbf{T})M(\mathbf{R}, \mathbf{P})R,$$

where T , P and R are the matrices derived from \mathbf{T} , \mathbf{P} and \mathbf{R} . From step (c) of TRANSUB, the tetrahedron \mathbf{T}_i^n is transformed from \mathbf{P}_i^n using $M(\mathbf{P}, \mathbf{T})$, so

$$T_i^n = M(\mathbf{P}, \mathbf{T})P_i^n.$$

With a suitable ordering of vertices of \mathbf{P}_i^n , $P_i^n = \alpha QP$ where α is a positive constant and Q is an orthogonal matrix, since \mathbf{P}_i^n is similar to \mathbf{P} by Lemma 1. So

$$T_i^n = \alpha M(\mathbf{P}, \mathbf{T})QP = \alpha M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P})R = M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P})R^n = M(\mathbf{R}^n, \mathbf{T}_i^n)R^n,$$

where \mathbf{R}^n is a regular tetrahedron having the same volume as \mathbf{T}_i^n .

From $\det(A(\mathbf{R}, \mathbf{T})) = \det(A(\mathbf{R}^n, \mathbf{T}_i^n)) = 1$ and (1), we have

$$(4) \quad \eta(\mathbf{T}) = \frac{3\sqrt[3]{\det(A(\mathbf{R}, \mathbf{T}))}}{\text{trace}(A(\mathbf{R}, \mathbf{T}))} = \frac{3}{\text{trace}((M(\mathbf{P}, \mathbf{T})M(\mathbf{R}, \mathbf{P}))^T M(\mathbf{P}, \mathbf{T})M(\mathbf{R}, \mathbf{P}))},$$

$$(5) \quad \eta(\mathbf{T}_i^n) = \frac{3\sqrt[3]{\det(A(\mathbf{R}^n, \mathbf{T}_i^n))}}{\text{trace}(A(\mathbf{R}^n, \mathbf{T}_i^n))} = \frac{3}{\text{trace}((M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P}))^T M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P}))}.$$

Let $\|B\|_F$ denote the Frobenius norm of matrix B [GoV89], i.e., $\|B\|_F = (\text{trace}(B^T B))^{1/2}$. By (4) and (5), it follows that

$$(6) \quad \frac{\eta(\mathbf{T}_i^n)}{\eta(\mathbf{T})} = \frac{\|M(\mathbf{P}, \mathbf{T})M(\mathbf{R}, \mathbf{P})\|_F^2}{\|M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P})\|_F^2}.$$

By the singular value decomposition [GoV89], $M(\mathbf{P}, \mathbf{T}) = Q_1 \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})Q_2$ where Q_1, Q_2 are orthogonal matrices and $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $A(\mathbf{P}, \mathbf{T})$ ($\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$ are the singular values of $M(\mathbf{P}, \mathbf{T})$). It follows that

$$(7) \quad \|M(\mathbf{P}, \mathbf{T})M(\mathbf{R}, \mathbf{P})\|_F^2 = \|\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})Q_2M(\mathbf{R}, \mathbf{P})\|_F^2,$$

$$(8) \quad \|M(\mathbf{P}, \mathbf{T})QM(\mathbf{R}, \mathbf{P})\|_F^2 = \|\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})Q_2QM(\mathbf{R}, \mathbf{P})\|_F^2.$$

Let $\mathbf{R}(\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ be a regular tetrahedron having the same volume as \mathbf{P} with coordinates $\mathbf{r}_0 = (-\sqrt{3}a/2, 0, 0)^T$, $\mathbf{r}_1 = (0, -a/2, 0)^T$, $\mathbf{r}_2 = (0, a/2, 0)^T$, and $\mathbf{r}_3 = (-\sqrt{3}a/6, 0, \sqrt{6}a/3)^T$. From the coordinates shown in Figure 3, with \mathbf{r}_i being transformed to \mathbf{p}_i , we have

$$M(\mathbf{R}, \mathbf{P}) = k \begin{bmatrix} 2\sqrt{2} & 0 & 1 \\ 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

where $k^3 = \sqrt{6}/72$. If U is an orthogonal matrix with elements u_{ij} , $1 \leq i, j \leq 3$, we have

$$\|\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})UM(\mathbf{R}, \mathbf{P})\|_F^2 =$$

$$k^2(\lambda_1[9 + 3(1 - (u_{11} - u_{13})^2)] + \lambda_2[9 + 3(1 - (u_{21} - u_{23})^2)] + \lambda_3[9 + 3(1 - (u_{31} - u_{33})^2)]).$$

Since $0 \leq (x + y)^2 \leq 2$ for any x, y satisfying $x^2 + y^2 \leq 1$,

$$6k^2(\lambda_1 + \lambda_2 + \lambda_3) \leq \|\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})UM(\mathbf{R}, \mathbf{P})\|_F^2 \leq 12k^2(\lambda_1 + \lambda_2 + \lambda_3).$$

Then combining (6), (7), and (8) yields

$$0.5 \leq \frac{\eta(\mathbf{T}_i^n)}{\eta(\mathbf{T})} \leq 2.$$

We now prove that the lower bound cannot be improved. For any $\epsilon > 0$, we define a tetrahedron \mathbf{T} with vertices $\mathbf{t}_0 = (0, \frac{\sqrt{11(0.5-\epsilon)}}{\sqrt{8\epsilon}}, 0)$, $\mathbf{t}_1 = (-1, 0, 0)$, $\mathbf{t}_2 = (1, 0, 0)$, and $\mathbf{t}_3 = (0, \frac{\sqrt{11(0.5-\epsilon)}}{\sqrt{8\epsilon}}, 1)$. Consider subtetrahedron $\mathbf{T}_1^1 = (\mathbf{t}_{03}, \mathbf{t}_{13}, \mathbf{t}_{12}, \mathbf{t}_{23})$ where \mathbf{t}_{ij} is the midpoint of \mathbf{t}_i and \mathbf{t}_j . By straightforward computation using (2), $\eta(\mathbf{T}_1^1)/\eta(\mathbf{T}) = 0.5 + \epsilon$. Hence the lower bound is tight. \square

In the subdivision procedure SUBDIV, the centre edge added in the interior of a tetrahedron is selected by a specific rule. Otherwise, refined subtetrahedra with good shape may not be guaranteed. In fact, there is a means of selecting centre edges such that the shape of some subtetrahedra become arbitrarily poor, as described by the following lemma (which is proved in [Liu94]) and theorem.

Lemma 2. *Suppose SUB_8 is applied to any tetrahedron $\mathbf{T}(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$. If the centre edge of \mathbf{T} is chosen such that the minimum η value of the four interior subtetrahedra achieves a minimum value among the three possible choices of centre edges, then there exists a subtetrahedron \mathbf{T}_1^1 of \mathbf{T} , such that $\eta(\mathbf{T}_1^1) \leq 6/7\eta(\mathbf{T})$, and the upper bound is tight.*

Theorem 4. *There exists a means of selecting the centre edges, such that after n levels of refinement by applying SUB_8 to \mathbf{T} ,*

$$\eta(\mathbf{T}_i^n) \leq (6/7)^n \eta(\mathbf{T}),$$

for some refined subtetrahedron \mathbf{T}_i^n .

Proof. At each step of refinement, if the centre edge is chosen as described in Lemma 2, the theorem can be established by repeatedly using Lemma 2. \square

In light of the scheme of selecting the centre edge in Lemma 2, we can always choose the centre edge such that the minimum shape measure of the four interior subtetrahedra achieves a maximum value, but it seems hard to obtain the nice properties in Theorems 2 and 3.

3. LOCAL REFINEMENT OF A CONFORMING MESH

Similar to the 2-D approach in [Ban90, p. 26], we design a 3-D LRA mainly depending on the regular subdivision SUB_8 illustrated in Figure 2, since the shape of refined tetrahedra produced by repeatedly performing SUB_8 to a tetrahedron are bounded below if each subtetrahedron has the same subdivision pattern as its parent by the discussion in the last section. Simply applying SUB_8 to a set of subtetrahedra of a conforming mesh may produce non-conformity between tetrahedra. Therefore, at the boundary of a refined region, other non-regular refinements may be needed to ensure a conforming mesh.

A *split point* is defined to be the midpoint of an edge whenever the edge needs subdivision. In an initial mesh \mathcal{T} , suppose a set S of tetrahedra are chosen for refinement. A split point is added to each edge of each tetrahedron in S . Also, for any face of a tetrahedron not in S that contains two split points, a split point is added to the edge that does not have a split point so that there are either one or three split points on each face. A face with 3 or 1 split point(s) is subdivided like Figure 1a or 1c, respectively; we don't use the subdivision of Figure 1b, since this will significantly increase the difficulty of keeping a conforming mesh. Under the above assumptions, the number of split points for a tetrahedron in the initial mesh can be 1, 2 (on a pair of opposite edges), 3 (on the same face), or 6. According to the number of split points, the subdivision of a tetrahedron is illustrated in Figures 4a, 4b, 4c, and Figure 2, respectively. The three non-regular refinements in Figures 4a, 4b, and 4c are denoted by SUB_2 , SUB_4^1 , and SUB_4^2 , respectively, and are applied to the boundary of a refined region. In the next refinement, if any tetrahedron, produced by SUB_2 , SUB_4^1 , or SUB_4^2 , is chosen for refinement, its parent is always refined by SUB_8 first. Thus we never need to add a split point on an edge, called an *irregular edge*, which is generated by connecting a vertex

to the midpoint of an edge of a face in SUB_2 , SUB_4^1 , or SUB_4^2 , e.g. $\mathbf{t}_3\mathbf{t}_{12}$ in Figure 4a and $\mathbf{t}_{03}\mathbf{t}_{12}$ in Figure 4b. An *irregular face* is defined to be one containing an irregular edge, e.g. face $\mathbf{t}_1\mathbf{t}_3\mathbf{t}_{12}$ in Figure 4a, and may have 0, 1, or 2 split points (for the next refinement). A face that is not irregular is called a *regular face*, and may have 0, 1, or 3 split points.

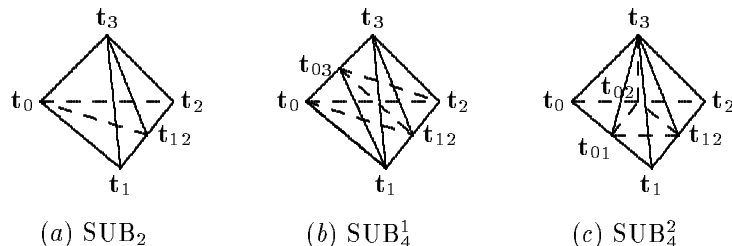


FIGURE 4. Illustration for non-regular refinements; $\mathbf{t}_{ij} = (\mathbf{t}_i + \mathbf{t}_j)/2$, $i < j$. (a) Subdivision of a tetrahedron with one split point (\mathbf{t}_{12}). (b) Subdivision of a tetrahedron with two split points (\mathbf{t}_{12} , \mathbf{t}_{03}) on a pair of opposite edges. (c) Subdivision of a tetrahedron with three split points (\mathbf{t}_{01} , \mathbf{t}_{02} , \mathbf{t}_{12}) on the same face.

We now give a local refinement procedure, QLRS (quality local refinement based on subdivision), based on SUB_8 , SUB_2 , SUB_4^1 , and SUB_4^2 . A tetrahedron \mathbf{T} is called a tetrahedron of type S_8 if \mathbf{T} is either a tetrahedron in the original mesh, or produced by SUB_8 . Similarly, a tetrahedron of type S_2 , S_4^1 , or S_4^2 means that the tetrahedron is produced by SUB_2 , SUB_4^1 , or SUB_4^2 , respectively. In the following, we assume that whenever SUB_8 is applied to a tetrahedron \mathbf{T} of the original mesh \mathcal{T} , the centre edge of \mathbf{T} is chosen such that the minimum shape value of the four interior subtetrahedra (i.e., \mathbf{T}_5^1 , \mathbf{T}_6^1 , \mathbf{T}_7^1 , \mathbf{T}_8^1 in Figure 2) of \mathbf{T} achieves a maximum value in terms of the tetrahedron shape measure η . Also, for any tetrahedron \mathbf{T} , an *unsplit edge* of \mathbf{T} is an edge that does not have a split point.

Algorithm QLRS

- (1) Set $\mathcal{T}^0 := \mathcal{T}$; $m := 0$.
- (2) Select a set S_m of tetrahedra needing refinement from \mathcal{T}^m ; for each tetrahedron \mathbf{T} in S_m , if \mathbf{T} is a tetrahedron of type S_8 , mark a split point on each unsplit edge of \mathbf{T} ; otherwise, mark a split point on each unsplit edge of \mathbf{T} 's parent.
- (3) For each tetrahedron \mathbf{T} of type S_2 in \mathcal{T}^m (see Figure 4a), whenever one of edges $\mathbf{t}_3\mathbf{t}_1$, $\mathbf{t}_3\mathbf{t}_2$ (or $\mathbf{t}_0\mathbf{t}_1$, $\mathbf{t}_0\mathbf{t}_2$) has a split point, mark a split point on the other if it does not have a split point; whenever $\mathbf{t}_1\mathbf{t}_{12}$ (or $\mathbf{t}_2\mathbf{t}_{12}$) has a split point, mark a split point on each unsplit edge of \mathbf{T} 's parent. For each tetrahedron of type S_4^1 or S_4^2 in \mathcal{T}^m , whenever one of its regular edges has a split point, mark a split point on each unsplit edge of \mathbf{T} 's parent. In addition, whenever a regular face in \mathcal{T}^m has two split points, mark a split point on the unsplit edge of the face; repeat the above process until no more split points are needed.
- (4) For any \mathbf{T} in \mathcal{T}^m with a non-zero number of split points, if \mathbf{T} is a tetrahedron of type S_8 , subdivide \mathbf{T} using SUB_8 , SUB_2 , SUB_4^1 , or SUB_4^2 according to the number of split points on \mathbf{T} , and let the subtetrahedra have the same subdivision pattern as \mathbf{T} ; otherwise, remove \mathbf{T} 's sibling(s), subdivide \mathbf{T} 's parent \mathbf{T}_p using SUB_8 , SUB_4^1 , or SUB_4^2 according to the number of split points on \mathbf{T}_p , and subdivide each subtetrahedron of \mathbf{T}_p if necessary using SUB_2 or

SUB_4^2 according to the number of split points on the subtetrahedron; label the conforming mesh by \mathcal{T}^{m+1} .

(5) $m := m + 1$; go to (2) if necessary or terminate.

Note that superscript m of \mathcal{T}^m indicates that mesh \mathcal{T}^m is created after m levels of refinement on \mathcal{T}^0 . In step (3) of QLRs, whenever a split point is added to an edge of a tetrahedron \mathbf{T} of type S_4^1 or S_4^2 , the split points are also added to the edges of \mathbf{T} 's parent \mathbf{T}_p . This guarantees that a tetrahedron of type S_4^1 or S_4^2 will never be further refined, i.e., whenever a tetrahedron \mathbf{T} of type S_4^1 or S_4^2 needs to be refined, \mathbf{T}_p is always first refined by the regular refinement. For a tetrahedron \mathbf{T} of type S_2 , its parent \mathbf{T}_p may be subdivided by SUB_4^1 , SUB_4^2 , or SUB_8 , and no subdivision is needed for subtetrahedra of \mathbf{T}_p if \mathbf{T}_p is subdivided by SUB_4^1 or SUB_4^2 . Figure 5 illustrates possible split points on subtetrahedra of \mathbf{T}_p after SUB_8 is applied to \mathbf{T}_p in step (4) of QLRs.

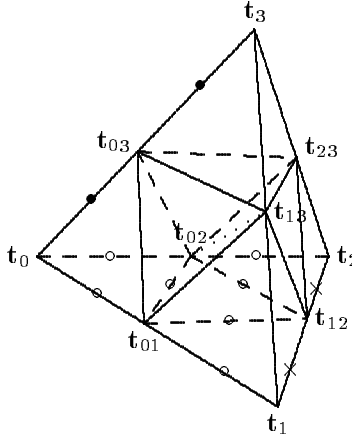


FIGURE 5. Illustration for possible split points after SUB_8 is applied to \mathbf{T} 's parent $\mathbf{T}_p(t_0, t_1, t_2, t_3)$. Case 1: \mathbf{T} is a tetrahedron of type S_2 (see Figure 4a); possible split points are on edges with \times . Case 2: \mathbf{T} is a tetrahedron of type S_4^1 (see Figure 4b); possible split points are on edges with \times or \bullet . Case 3: \mathbf{T} is a tetrahedron of type S_4^2 (see Figure 4c); possible split points are on edges with \times or \circ .

It is obvious that step (3) of QLRs will terminate in a finite number of steps, since in the worst case each edge, which is not an irregular edge, of \mathcal{T}^m has a split point. To guarantee the validity of QLRs, we need to prove that step (4) of QLRs produces a conforming mesh.

Theorem 5. *If \mathcal{T}^m is a conforming mesh, then step (4) of Algorithm QLRs produces a conforming mesh \mathcal{T}^{m+1} .*

Proof. Let \mathbf{T} be a tetrahedron of type S_2 , S_4^1 , or S_4^2 in \mathcal{T}^m , i.e., \mathbf{T} 's parent \mathbf{T}_p has the configuration of Figure 4a, 4b, or 4c (we assume that \mathbf{T}_p has tetrahedron vertices t_0, t_1, t_2 , and t_3). We first prove that if SUB_8 is applied to \mathbf{T}_p in step (4) of QLRs, each face of a subtetrahedron of \mathbf{T}_p has 0, 1, or 3 split points. Note that among edges of \mathbf{T} or \mathbf{T} 's sibling(s), only edges t_1t_{12}, t_2t_{12} of Figure 4a (see Case 1 in Figure 5); $t_0t_{03}, t_3t_{03}, t_1t_{12}, t_2t_{12}$ of Figure 4b (see Case 2 in Figure 5); and all 9 edges on face $t_0t_1t_2$ of Figure 4c (see Case 3 in Figure 5) exist in subtetrahedra produced by applying SUB_8 to \mathbf{T}_p . It can be easily verified that each face of a subtetrahedron of \mathbf{T}_p has 0, 1, or 3 edges of \mathcal{T}^m , and no pair of opposite edges in a subtetrahedron of \mathbf{T}_p are both in \mathcal{T}^m . This guarantees that no extra split point is needed for subtetrahedra of \mathbf{T}_p in order to satisfy that each

face of a subtetrahedron does not have 2 split points, since only edges of \mathcal{T}^m may have split points. Therefore, it suffices to apply SUB₂ or SUB₄² to subtetrahedra of \mathbf{T}_p in step (4) of QLRs.

Now we prove that step (4) of QLRs produces a conforming mesh. It is obvious that any two tetrahedra cannot intersect in their interior. Therefore it suffices to prove that the resulting mesh is conforming on any face. Suppose F is an interior face in a conforming mesh \mathcal{T}^m , $m \geq 0$. If F is a regular face, it will have the configuration of Figure 1a, 1c, or F itself at the end of step (4) of QLRs depending on the number of split points on the face, independent of the use of SUB₂, SUB₄¹, SUB₄², or SUB₈ on the tetrahedra sharing F . If F is an irregular face, by step (4) of QLRs, either F does not change (in this case, the conformity is obvious) or the parents of the tetrahedra sharing F are first subdivided by SUB₈ or SUB₄² without producing non-conformity on F 's direct parent F' , i.e. F' is subdivided into the configuration of Figure 1a. Further possible subdivisions on the subfaces of F' can be treated just like the case for regular face, since the subfaces of F' are regular. Thus \mathcal{T}^{m+1} is conforming. \square

Since we mainly use SUB₈ in QLRs, Theorem 6 below gives a property similar to that in Theorem 2. A numerical bound on the shape of subtetrahedra produced by QLRs in terms of tetrahedron shape measure η is given in Theorem 7, which needs Lemma 3.

Theorem 6. *For any initial mesh \mathcal{T} , there are only a finite number, which depends on the number of tetrahedra in \mathcal{T} , of classes of similar tetrahedra in all refined tetrahedra generated by QLRs.*

Proof. Note that in QLRs, if a tetrahedron is subdivided to $n \geq 1$ levels, SUB₈ is applied in the first $n - 1$ levels of refinement, and each subtetrahedron is set to have the same subdivision pattern as its parent. At the n th level of refinement, SUB₈, SUB₂, SUB₄¹, or SUB₄² is used. The theorem is established by Theorem 2, since each class of similar tetrahedra in the first $n - 1$ levels of refinement produces at most 4 new classes of similar tetrahedra at the last step of refinement. \square

Lemma 3. *Let \mathbf{T}_k^1 be any subtetrahedron generated by SUB₂, SUB₄¹, or SUB₄² applied to \mathbf{T} . Then*

$$(9) \quad \eta(\mathbf{T}_k^1) \geq 2\sqrt[3]{4}/11\eta(\mathbf{T}).$$

Proof. Let \mathbf{t}_{ij} denote the midpoint of $\mathbf{t}_i\mathbf{t}_j$ and $l_{ij} = |\mathbf{t}_i\mathbf{t}_j|$. For a triangle $\mathbf{t}_A\mathbf{t}_B\mathbf{t}_C$,

$$(10) \quad |\mathbf{t}_C\mathbf{t}_{AB}|^2 = (2l_{AC}^2 + 2l_{BC}^2 - l_{AB}^2)/4,$$

and for a tetrahedron with vertices $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$,

$$(11) \quad |\mathbf{t}_{03}\mathbf{t}_{12}|^2 = (\mathbf{t}_0 - \mathbf{t}_1 + \mathbf{t}_3 - \mathbf{t}_2)^T(\mathbf{t}_0 - \mathbf{t}_1 + \mathbf{t}_3 - \mathbf{t}_2)/4 = (l_{01}^2 + l_{23}^2 + l_{02}^2 + l_{13}^2 - l_{03}^2 - l_{12}^2)/4.$$

Case 1: \mathbf{T}_k^1 is a subtetrahedron generated by SUB₂ (see Figure 4a). By simple algebraic manipulation using (10), (2), and the fact that the volume of \mathbf{T}_k^1 is half the volume of \mathbf{T} , it follows that

$$(12) \quad \frac{\eta(\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_{12}, \mathbf{t}_3))}{\eta(\mathbf{T})} = 2\sqrt[3]{2} \times \frac{\sum_{0 \leq i < j \leq 3} l_{ij}^2}{6l_{01}^2 + 6l_{13}^2 + 4l_{03}^2 + 2l_{23}^2 + 2l_{02}^2 - l_{12}^2} \geq \sqrt[3]{2}/3.$$

Similarly, we obtain

$$(13) \quad \eta(\mathbf{T}_2^1(\mathbf{t}_0, \mathbf{t}_2, \mathbf{t}_{12}, \mathbf{t}_3)) \geq \sqrt[3]{2}/3\eta(\mathbf{T}).$$

Case 2: \mathbf{T}_k^1 is a subtetrahedron generated by SUB₄¹ (see Figure 4b). Using (10), (11), and (2),

$$(14) \quad \frac{\eta(\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_{12}, \mathbf{t}_{03}))}{\eta(\mathbf{T})} = \sqrt[3]{4} \times \frac{\sum_{0 \leq i < j \leq 3} l_{ij}^2}{9l_{01}^2 + 3l_{13}^2 + 3l_{02}^2 + l_{23}^2 - l_{03}^2 - l_{12}^2}.$$

By the triangle inequality,

$$(15) \quad l_{01}^2 \leq 2(l_{03}^2 + l_{13}^2) \leq 2(l_{03}^2 + 2l_{12}^2 + 2l_{23}^2),$$

$$(16) \quad l_{01}^2 \leq l_{02}^2 + l_{03}^2 + l_{12}^2 + l_{13}^2.$$

Note that $9l_{01}^2 = l_{01}^2 + 5l_{01}^2/2 + 11l_{01}^2/2$; if the first two items on the right side of this equality are replaced by (15) and (16), respectively, it follows that the denominator of the right side of (14) is $\leq 11l_{01}^2/2 + 11l_{13}^2/2 + 11l_{02}^2/2 + 11l_{12}^2/2 + 5l_{23}^2 + 7l_{03}^2/2$. Hence

$$(17) \quad \frac{\eta(\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_{12}, \mathbf{t}_{03}))}{\eta(\mathbf{T})} \geq \frac{2\sqrt[3]{4}}{11}.$$

Similarly, for the other three subtetrahedra, we obtain the same bound as in (17).

Case 3: \mathbf{T}_k^1 is a subtetrahedron generated by SUB_4^2 (see Figure 4c). Using similar techniques as above, we obtain

$$(18) \quad \frac{\eta(\mathbf{T}_1^1(\mathbf{t}_0, \mathbf{t}_{01}, \mathbf{t}_{02}, \mathbf{t}_3))}{\eta(\mathbf{T})} = \sqrt[3]{4} \times \frac{\sum_{0 \leq i < j \leq 3} l_{ij}^2}{8l_{03}^2 + 2l_{13}^2 + 2l_{23}^2 + l_{12}^2} \geq \frac{\sqrt[3]{4}}{5},$$

since $3l_{03}^2 \leq 3l_{01}^2 + 3l_{02}^2 + 3l_{13}^2 + 3l_{23}^2$. For the other two subtetrahedra in the corners, we obtain the same bound as in (18). For the subtetrahedron in the middle, we have,

$$(19) \quad \frac{\eta(\mathbf{T}_4^1(\mathbf{t}_{01}, \mathbf{t}_{12}, \mathbf{t}_{02}, \mathbf{t}_3))}{\eta(\mathbf{T})} = \sqrt[3]{4} \times \frac{\sum_{0 \leq i < j \leq 3} l_{ij}^2}{4l_{03}^2 + 4l_{13}^2 + 4l_{23}^2} \geq \frac{\sqrt[3]{4}}{4}.$$

Combining (12), (13), (17), (18), and (19) yields (9). \square

Theorem 7. *If \mathbf{T}_i^n is a refined tetrahedron of any tetrahedron \mathbf{T} in \mathcal{T} , produced by QLRs, then*

$$(20) \quad \eta(\mathbf{T}_i^n) \geq c \eta(\mathbf{T}),$$

where $c = \sqrt[3]{4}/11 \approx 0.1443$.

Proof. If \mathbf{T}_i^n is of type S_8 , then (20) follows from (3). Suppose \mathbf{T}_i^n is of type S_2 , S_4^1 , or S_4^2 , and \mathbf{T}_j^{n-1} is the direct parent of \mathbf{T}_i^n , $n \geq 1$. According to QLRs, \mathbf{T}_j^{n-1} is produced by repeatedly applying SUB_8 to \mathbf{T} . By Theorem 3,

$$(21) \quad \eta(\mathbf{T}_j^{n-1}) \geq 0.5\eta(\mathbf{T}).$$

By Lemma 3,

$$(22) \quad \eta(\mathbf{T}_i^n) \geq 2\sqrt[3]{4}/11 \eta(\mathbf{T}_j^{n-1}).$$

Combining (21) and (22) yields (20). \square

For any tetrahedron \mathbf{T} in \mathcal{T}^m , $m \geq 0$, we recursively define the *subdivision level* of \mathbf{T} , denoted by $\ell(\mathbf{T})$, as follows. If \mathbf{T} is a tetrahedron of the initial mesh \mathcal{T} , $\ell(\mathbf{T}) = 0$; otherwise, $\ell(\mathbf{T}) = \ell(\mathbf{T}_p) + 3$, $\ell(\mathbf{T}_p) + 2$, or $\ell(\mathbf{T}_p) + 1$ if \mathbf{T} is a subtetrahedron produced by applying SUB_8 , SUB_4^1 or SUB_4^2 , or SUB_2 to \mathbf{T}_p , respectively.

Theorem 8. *For any interior face shared by two adjacent tetrahedra \mathbf{T}_L and \mathbf{T}_R in \mathcal{T}^m , $m \geq 0$, generated by QLRs, $|\ell(\mathbf{T}_L) - \ell(\mathbf{T}_R)| \leq 2$, and the upper bound is tight.*

Proof. Let F be an interior face shared by two adjacent tetrahedra \mathbf{T}_L and \mathbf{T}_R in \mathcal{T}^m , $m \geq 0$. We first prove by contradiction that if \mathbf{T}_L and \mathbf{T}_R are tetrahedra of type S_8 , $\ell(\mathbf{T}_L) = \ell(\mathbf{T}_R)$. Let $\ell(\mathbf{T}_L) = 3p$, $\ell(\mathbf{T}_R) = 3q < 3p$, and $0 < r \leq q$. Note that when SUB_8 is applied to a tetrahedron, each face of the tetrahedron is subdivided. Therefore, for any r , the ancestor $\hat{\mathbf{T}}_L$ of \mathbf{T}_L at level $3p - 3r$ must have a common face with the ancestor $\hat{\mathbf{T}}_R$ of \mathbf{T}_R at level $3q - 3r$. Let $r = q$. Then $\hat{\mathbf{T}}_R$ is a tetrahedron in the initial mesh, but $\hat{\mathbf{T}}_L$ is a tetrahedron at level $3(p - q)$. These two tetrahedra cannot share a common face, a contradiction.

Suppose one of the two tetrahedra, \mathbf{T}_L say, is not a tetrahedron of type S_8 . If \mathbf{T}_R is a tetrahedron of type S_8 , then F must be like face $\mathbf{t}_0\mathbf{t}_1\mathbf{t}_3$ (or $\mathbf{t}_0\mathbf{t}_2\mathbf{t}_3$) in Figure 4a, or a subface of $\mathbf{t}_0\mathbf{t}_1\mathbf{t}_2$ in Figure 4c. In the former case, $\ell(\mathbf{T}_R) = \ell(\mathbf{T}_L) - 1$. In the latter case, $\ell(\mathbf{T}_R) = \ell(\mathbf{T}_L) + 1$.

Now suppose \mathbf{T}_R is also not a tetrahedron of type S_8 . First, if F is an irregular face, $\ell(\mathbf{T}_L) = \ell(\hat{\mathbf{T}}_L) + 1$ or $\ell(\hat{\mathbf{T}}_L) + 2$, and $\ell(\mathbf{T}_R) = \ell(\hat{\mathbf{T}}_R) + 1$ or $\ell(\hat{\mathbf{T}}_R) + 2$, where $\hat{\mathbf{T}}_L$ and $\hat{\mathbf{T}}_R$ are \mathbf{T}_L and \mathbf{T}_R 's parents, respectively. So, $|\ell(\mathbf{T}_L) - \ell(\mathbf{T}_R)| \leq 1$, since $\ell(\hat{\mathbf{T}}_R) = \ell(\hat{\mathbf{T}}_L)$ by the discussion in the first paragraph. Next, if F is a regular face, the types of \mathbf{T}_L and \mathbf{T}_R can be one of the following pairs (S_2, S_2) , (S_4^2, S_4^2) , and (S_2, S_4^2) . For the first two pairs, $\ell(\mathbf{T}_L) = \ell(\mathbf{T}_R)$. For the final pair, suppose \mathbf{T}_L is a tetrahedron of type S_2 , and \mathbf{T}_R is a tetrahedron of type S_4^2 . Then the only possible situation is that the parent $\hat{\mathbf{T}}_R$ of \mathbf{T}_R has a common face with the direct grandparent $\hat{\mathbf{T}}_L$ of \mathbf{T}_L . Since $\ell(\mathbf{T}_R) = \ell(\hat{\mathbf{T}}_R) + 2$, $\ell(\mathbf{T}_L) = \ell(\hat{\mathbf{T}}_L) + 3 + 1$, and $\ell(\hat{\mathbf{T}}_R) = \ell(\hat{\mathbf{T}}_L)$, it follows that $\ell(\mathbf{T}_L) = \ell(\mathbf{T}_R) + 2$. \square

Note that if the maximum subdivision level of a tetrahedron in \mathcal{T}^k is $\leq 3k$, then the maximum subdivision level of a tetrahedron in \mathcal{T}^{k+1} is $\leq 3(k+1)$, since each subtetrahedron of a tetrahedron of type S_8 in \mathcal{T}^k has subdivision level at most $3k+3$ in \mathcal{T}^{k+1} , and each subtetrahedron of the parent of a tetrahedron of type S_2 , S_4^1 , or S_4^2 in \mathcal{T}^k has subdivision level at most $3(k-1) + 3 + 2$ in \mathcal{T}^{k+1} . So, by induction on k , the maximum subdivision level of a tetrahedron in any \mathcal{T}^m is $\leq 3m$, $m \geq 0$.

4. EXPERIMENTAL RESULTS

In QLRS, the main operation is to determine the number of split points for each tetrahedron subject to the condition that each regular face has 0, 1, or 3 split points and each irregular face has 0, 1, or 2 split points. When a split point is added to an edge e , each regular face incident on e is checked to decide whether an extra split point is needed for this face. Therefore, the data structures for QLRS should support a fast operation that reports all faces incident on an edge, which can be done by recording the adjacent tetrahedron for each interior face of a tetrahedron. In addition, determining whether an edge has a split point should also be implemented efficiently. To this end, we have designed a set of data structures similar to those in [LiJ94c], in which a hash table and a stack are used to store all edges with a split point and all tetrahedra needing subdivision at each step of refinement, respectively, and have implemented QLRS in an expected time complexity that is linearly related to the number of refined tetrahedra in a refined mesh (see details in [Liu94]).

We report our experimental results for four single tetrahedra (Tables I-IV in [RiL92]) and two tetrahedral meshes of polyhedral regions (one is a convex polyhedron, Figure 1a in [Joe91]; the other is a U-shaped region, Figure 10 in [Joe94]). For a single tetrahedron, we refine all tetrahedra in the mesh at each step of refinement as in [RiL92, LiJ94c]. For the two tetrahedral meshes, a fixed point on the object is chosen as the center of a sphere; at each step of refinement, we refine any tetrahedron with at least one of its vertices in the sphere, and reduce the radius of the sphere by a factor. The quantities used to measure the quality at each level of refinement are the number of tetrahedra NTET, the minimum mean ratio η_{min} (over all tetrahedra), the average mean ratio η_{ave} , and the percentage of tetrahedra whose η value is less/greater than a certain number. The

mean ratio η ($0 < \eta \leq 1$) approaches zero or the maximum value for a poorly-shaped or well-shaped tetrahedron, respectively. In our experiments, a tetrahedron with $\eta < 0.3$ (or $\eta \geq 0.7$) is considered poorly-shaped (or well-shaped). The reason for using η to report our experimental results is because our theoretical results are based on it, and we don't know whether Theorems 3 and 7 hold for other (valid) tetrahedron shape measures. By the "equivalence" of tetrahedron shape measures [LiJ94b], we can expect similar statistical results based on other tetrahedron shape measures.

The four single tetrahedra are listed in Table 1 in terms of the coordinates of the four vertices. P1 and P2 are well-shaped tetrahedra; P3 is a poorly-shaped tetrahedron; P4 is the regular tetrahedron, where $\sqrt{3}$ and $\sqrt{2}$ are rounded to 16 decimal places. For the two polyhedral regions, by using the methods described in [Joe91, Joe94], the convex polyhedron is subdivided into 273 tetrahedra, and has minimum mean ratio 0.6230 after local transformations are used to obtain an improved-quality mesh with respect to radius ratio ρ ; the U-shaped object is subdivided into 466 tetrahedra with minimum mean ratio 0.5580 after local transformations are performed.

Tables 2 to 4 show experimental results based on QLRs. The results seem to confirm that η_{min} and η_{ave} converge asymptotically to a fixed value for QLRs, and the experimental results are much better than the theoretical estimate of the constant c in Theorem 7 (since we do not have a tight bound).

Table 1: Problems 1 to 4

P1			P2			P3			P4		
$\eta = 0.8846$			$\eta = 0.8399$			$\eta = 0.2835$			$\eta = 1.0000$		
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4.0	2.0	2.0	4.0	0.0	0.0	0.5	0.0	0.0	$2\sqrt{3}$	0.0	0.0
1.0	5.0	0.0	0.0	4.0	0.0	1.5	5.0	2.0	$\sqrt{3}$	3.0	0.0
0.5	0.5	5.0	0.0	0.0	4.0	0.5	0.5	5.0	$\sqrt{3}$	1.0	$2\sqrt{2}$

Table 2: Experimental results for Problems 1, 2

level	Problem 1					Problem 2				
	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$
0	1	0.8846	0.8846	0.00	100.00	1	0.8399	0.8399	0.00	100.00
1	8	0.8664	0.9069	0.00	100.00	8	0.6872	0.7808	0.00	75.00
2	64	0.8664	0.9124	0.00	100.00	64	0.6872	0.7660	0.00	68.75
3	512	0.8664	0.9138	0.00	100.00	512	0.6872	0.7623	0.00	67.19

Table 3: Experimental results for Problems 3, 4

level	Problem 3					Problem 4				
	NTET	η_{min}	η_{ave}	$\eta < 0.2$	$\eta \geq 0.4$	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$
0	1	0.2835	0.2835	0.00	0.00	1	1.0000	1.0000	0.00	100.00
1	8	0.2756	0.2819	0.00	0.00	8	0.8571	0.9286	0.00	100.00
2	64	0.2756	0.2815	0.00	0.00	64	0.8571	0.9107	0.00	100.00
3	512	0.2756	0.2814	0.00	0.00	512	0.8571	0.9062	0.00	100.00

Table 4: Experimental results for two tetrahedral meshes

level	Convex Polyhedron					U-shaped Object				
	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$
0	273	0.6230	0.8185	0.00	92.67	466	0.5580	0.7651	0.00	80.47
1	419	0.4265	0.7790	2.63	80.91	588	0.4373	0.7628	1.19	79.25
2	843	0.3729	0.7611	5.10	79.12	931	0.3957	0.7514	3.33	74.33
3	1392	0.3729	0.7616	4.24	78.38	1494	0.3877	0.7430	4.95	73.09
4	1830	0.3729	0.7611	3.88	77.54	2020	0.3877	0.7378	5.74	72.13

Let NTETC and NTETR denote the number of tetrahedra chosen for refinement and the number of tetrahedra actually refined, respectively, at each step of refinement. The ratio NTETR/NTETC reflects the expansion of refinement at a local region. Table 5 lists NTETC, NTETR, NTETR/NTETC, and NTET for the two polyhedral regions. At each step of refinement, we choose a small number of tetrahedra around a fixed point for refinement. It seems that the ratio NTETR/NTETC is bounded above by a small constant (i.e., NTETR does not expand rapidly against NTETC), which is what we expect for “local” refinement.

Table 5: Performance on the expansion of refinement

	Convex Polyhedron					U-shaped Object				
NTETC	15	38	62	47	24	14	39	64	57	41
NTETR	38	116	121	100	54	26	76	123	116	87
NTETR/NTETC	2.53	3.05	1.95	2.13	2.25	1.86	1.95	1.92	2.04	2.12
NTET	273	419	843	1392	1830	466	588	931	1494	2020

Similar to [LiJ94c], local transformations [Joe89], which are based on two different triangulations of certain configurations of five distinct non-coplanar 3-D points, can be applied with respect to some criteria, such as locally improving the minimum tetrahedron shape measure value. The algorithms in [Joe89, Joe93] can be used to improve the quality of tetrahedra towards an optimal mesh with respect to η . To keep the properties in Theorems 6 and 7, we only apply local transformations to the final refined mesh, i.e., no refinement is applied to meshes improved by local transformations. The statistical results for the improved-quality meshes of the two polyhedral regions are provided in Table 6. The improvements are not very significant, compared with the improved results in [LiJ94c]. This may imply that QLRS produces tetrahedra of relatively good shape, compared with local refinement based on bisection.

Table 6: Improved-quality meshes for two polyhedral regions

level	Convex Polyhedron					U-shaped Object				
	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$	NTET	η_{min}	η_{ave}	$\eta < 0.5$	$\eta \geq 0.7$
0	271	0.6230	0.8203	0.00	92.62	464	0.5825	0.7664	0.00	80.17
1	415	0.4297	0.7804	2.41	82.41	586	0.5554	0.7688	0.00	79.86
2	838	0.4297	0.7735	2.15	81.62	929	0.4458	0.7611	0.75	76.21
3	1388	0.4297	0.7710	1.51	80.04	1491	0.4348	0.7536	1.27	73.91
4	1825	0.4297	0.7700	1.15	79.01	2014	0.4159	0.7478	1.84	72.44

5. CONCLUDING REMARKS

We have presented a local refinement algorithm based mainly on an 8-subtetrahedron subdivision, and have shown that the algorithm produces guaranteed-quality meshes. Experimental results show that the number of tetrahedra actually refined in order to keep a conforming mesh is limited

by a small number times the number of tetrahedra chosen for refinement, which implies that the algorithm is truly local. The quality of refined meshes for various examples show a very consistent and satisfactory performance.

The algorithm presented here is preferable if the mesh in a refined region is relatively coarse; otherwise, the one described in [LiJ94c] based on a bisection procedure may be better. It is worth considering the situation that QLRs is applied to some parts of a refined region while the algorithm in [LiJ94c] is used in other parts. A critical problem is to ensure a conforming mesh with guaranteed-quality when the expansion of refinement from different parts meet each another.

REFERENCES

- [Ban90] R. E. Bank (1990), *PLTMG: A Software Package for Solving Elliptic Partial Differential Equations: Users' Guide 6.0*, SIAM, Philadelphia.
- [BaS81] R. E. Bank and A. H. Sherman (1981), An adaptive multilevel method for elliptic boundary value problems, *Computing*, 29, pp. 91-105.
- [GoV89] G.H. Golub and C.F. Van Loan (1989), *Matrix Computations*, 2nd edition, Johns Hopkins University Press, Baltimore.
- [Joe89] B. Joe (1989), Three-dimensional triangulations from local transformations, *SIAM J. Sci. Stat. Comput.*, 10, pp. 718-741.
- [Joe91] B. Joe (1991), Delaunay versus max-min solid angle triangulations for three-dimensional mesh generation, *Intern. J. Num. Meth. Eng.*, 31, pp. 987-997.
- [Joe93] B. Joe (1993), Construction of three-dimensional improved-quality triangulations using local transformations, submitted for publication.
- [Joe94] B. Joe (1994), Tetrahedral mesh generation in polyhedral regions based on convex polyhedron decompositions, *Intern. J. Num. Meth. Eng.*, 37, pp. 693-713.
- [LiJ94a] A. Liu and B. Joe (1994), On the shape of tetrahedra from bisection, to appear in *Mathematics of Computation*.
- [LiJ94b] A. Liu and B. Joe (1994), Relationship between tetrahedron shape measures, to appear in *BIT*.
- [LiJ94c] A. Liu and B. Joe (1994), Quality local refinement of tetrahedral meshes based on bisection, submitted for publication.
- [Liu94] A. Liu (1994), Quality local refinement of tetrahedral meshes, *Ph.D. Thesis, Department of Computing Science, University of Alberta*.
- [LMZ85] R. Löhner, K. Morgan, and O. C. Zienkiewicz (1985), Adaptive grid refinement for the compressible Euler equations, in I. Babuska et al. *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*, Wiley, New York.
- [Mit89] W. F. Mitchell (1989), A comparison of adaptive refinement techniques for elliptic problems, *ACM Trans. on Mathematical Software*, 15, pp. 326-347.
- [MoW90] D. Moore and J. Warren (1990), Adaptive mesh generation I: packing space, *Technical Report, Department of Computer Science, Rice University*
- [NeF91] T. W. Nehl and D. A. Field (1991), Adaptive refinement of first order tetrahedral meshes for magnetostatics using local Delaunay subdivisions, *IEEE Trans. on Magnetics*, 27, pp. 4193-4196.
- [Riv84] M.-C. Rivara (1984), Algorithms for refining triangular grids suitable for adaptive and multigrid techniques, *Intern. J. Num. Meth. Eng.*, 20, pp. 745-756.
- [Riv87] M.-C. Rivara (1987), A grid generator based on 4-triangles conforming mesh-refinement algorithms, *Intern. J. Num. Meth. Eng.*, 24, pp. 1343-1354.
- [Riv89] M.-C. Rivara (1989), Selective refinement/derefinement algorithms for sequences of nested triangulations, *Intern. J. Num. Meth. Eng.*, 28, pp. 2889-2906.
- [RiL92] M.-C. Rivara and C. Levin (1992), A 3-D refinement algorithm suitable for adaptive and multi-grid techniques, *Comm. Appl. Num. Meth.*, 8, pp. 281-290.